



**NATIONAL OPEN UNIVERSITY OF NIGERIA**  
**University Village, Plot 91, Cadastral Zone, Nnamdi Azikwe Express Way, Jabi-Abuja**  
**FACULTY OF SCIENCES**  
**DEPARTMENT OF MATHEMATICS**  
**2025\_2 EXAMINATIONS**

---

**Course Title: Functional Analysis II**

**Credit Unit: 3**

**Time Allowed: 3 Hours**

**Total: 70 Marks**

**Instruction: Answer Question One (1) and Any Other Three (3) Questions**

**Question One**

**1(a)** Let  $X = \mathbb{R}^2$ . For arbitrary  $\bar{x} = (x_1, x_2)$  in  $X$ , define the function  $\|\cdot\|_2: \mathbb{R}^2 \rightarrow [0, \infty)$  by  $\|\cdot\|_2 = (x_1^2 + x_2^2)^{1/2}$  by only showing that this function satisfies the triangle inequality, prove that it defines a norm on  $\mathbb{R}^2$

**(b)** Let  $X = \mathbb{R}$ , the set of real numbers. Show that the map  $\rho(x, y) = |x - y|$  defines a metric on  $\mathbb{R}$ .

**(c)** Every normed space is automatically a metric space. Prove this claim

**(d)(i)** What is a Banach space?

**(ii)** What does it mean to say a normed space is complete?

**(e) (i)** All normed spaces are metric spaces. Does the converse of this statement hold?

**(ii)** Let  $X = l_2 := \{x = (x_1, x_2, \dots) : x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ . Let  $x = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ .

Is  $x \in l_2$

**(25marks)**

**Question Two**

**(a)** Prove that the space  $C[a, b]$  of continuous real valued functions defined on the closed and bounded interval  $[a, b]$  is complete if endowed with the sup norm  $\|f\|_0 = \max_{a \leq t \leq b} |f(t)|$  for arbitrary  $f \in C[a, b]$

**(b)** Let  $(X, \rho)$  be a complete metric space, and let  $E \subset X$ .  $(E, \rho_E)$  is complete if and only if it is closed. Where  $\rho_E$  is the subspace metric induced by  $\rho$ . Prove this theorem. **(15marks)**

**Question Three**

**(a)** Let  $X$  be a linear space. Define a map  $T: X \times X \rightarrow X \times X$  by

$$T(x, y) = (x + 2y, x - 2y) \quad ; \quad (x, y) \in X \times X$$

**(i)** Determine if  $T$  is linear **(ii)** Does  $T$  have an inverse map?

**(b)** Let  $\mathbb{C}$  denote the linear space of complex numbers over  $\mathbb{C}$  (as the field of scalars), and define the map  $T: \mathbb{C} \rightarrow \mathbb{C}$  by  $Tz = \bar{z}$ , where  $\bar{z}$  denotes the complex conjugate of  $z$ . Prove that  $T$  is not a linear map. **(15marks)**

#### Question Four

**(a)** Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space. Prove that the function  $\|\cdot\|: X \rightarrow \mathbb{R}$  defined by  $\|x\| = \sqrt{\langle x, x \rangle}$  is a norm on  $X$ .

**B (i)** Given an inner product space  $(X, \langle \cdot, \cdot \rangle)$ , Prove the equality

$$\langle z, \lambda x + \mu y \rangle = \bar{\lambda} \langle z, x \rangle + \bar{\mu} \langle z, y \rangle \text{ for arbitrary vectors } x, y, z \in X \text{ and scalars } \lambda, \mu \in \mathbb{C}$$

**(ii)** What is the usefulness of the parallelogram law?

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) ? \text{ (15marks)}$$

#### Question Five

**(a)(i)** What is a Hilbert space? **(ii)** Given an inner product space  $(X, \langle \cdot, \cdot \rangle)$  and a set  $S \subset X$ , when is  $S$  said to be Orthonormal? **(iii)** What is a complete orthonormal set?

**(b)** Let  $\mathbb{H}$  be a real Hilbert space and let  $M$  be a closed subspace of  $\mathbb{H}$ . Consider an arbitrary vector  $x \in \mathbb{H}$ . State three major questions answered by the projection mapping theorem

**(c)(i)** Let  $M$  and  $N$  be arbitrary subspaces of a Hilbert space  $\mathbb{H}$ .

Prove the following  $M \subset M^{\perp\perp}$

**(ii)** State the Riesz Representation Theorem **(15marks)**

#### Question Six

**(a)** Let  $\mathbb{H}$  be a Hilbert space. When is an operator  $T: \mathbb{H} \rightarrow \mathbb{H}$  said to be

**(i)** Self-adjoint **(ii)** Normal **(iii)** Unitary

**(b)** Consider the right shift operator on  $l_2$ ,  $T: l_2 \rightarrow l_2$  defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

**(i)** Find the adjoint of this map **(ii)** Is this map normal? **(15marks)**