



NATIONAL OPEN UNIVERSITY OF NIGERIA  
Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway, Jabi, Abuja.

FACULTY OF SCIENCES  
DEPARTMENT OF MATHEMATICS  
September Examination 2020

Course Code: MTH 422  
Course Title: Partial Differential Equations  
Credit Unit: 3  
Time Allowed: 3 Hours  
Instruction: Answer Question Number One (1) and any Other Four (4) Questions

1. (a) Give any four classical examples of partial Differential equations (1 marks each)
- (b). In the following decide whether the given partial differential equations and boundary or initial conditions are linear or nonlinear, and, if linear, whether they are homogeneous or nonhomogeneous. Determine the order of the partial differential equation.
- i.  $u_{xx} + u_{xy} = 2u, u_x(0, y) = 0$
- ii.  $u_{xx} + xu_{xy} = 2, u(x, 0) = 0, u(x, 1) = 0$
- iii.  $u_{xx} - u_t = f(x, t), u_t(x, 0) = 2$  (3 marks each)
- (c). Verify that the given functions are solutions of two dimensional Laplace equation
- i.  $u = x + y$
- ii.  $u = x^2 - y^2$
- iii.  $u = \frac{x}{x^2 + y^2}$  (3marks each)
2. (a). Derive the general solution of the following equations by using an appropriate change of variables
- i.  $\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$  (2 marks)
- ii.  $2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$  (2 marks)
- (b). Suppose that  $u = u(x, t)$  and  $v = v(x, t)$  have partial derivatives related in the following way:
- $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} = -\frac{\partial v}{\partial x}$$
- Show that  $u$  and  $v$  are solution of the wave equation  $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$  with  $c = 1$  (3 marks)
- (c). Solve the boundary value problem  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$   $0 < x < L, t > 0$ , for a string of unit length, subject to the given conditions.  $f(x) = .05 \sin \pi x, g(x) = 0, c = \frac{1}{\pi}$  (5 marks)
3. a. Consider the equations  $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial t} = -\frac{\partial v}{\partial x}$  supplemented by the initial data  $u(x, 0) = f(x), v(x, 0) = h(x)$

(i) Show that the appropriate initial data for the wave equation for  $u$  is

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = -h'(x) \quad (2 \text{ marks})$$

(ii) Find the appropriate initial data for the wave equation for  $v$ . (2 marks)

b. Use the change of variables  $\alpha = x + ct$ ,  $\beta = x - ct$  to transform the wave equation  $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$  into  $\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0$ . (you should assume that  $\frac{\partial^2 u}{\partial \alpha \partial \beta} = \frac{\partial^2 u}{\partial \beta \partial \alpha}$ .) (4 marks)

c. Integrate the equation with respect to  $\alpha$  to obtain  $\frac{\partial u}{\partial \beta} = g(\beta)$ , where  $g$  is an arbitrary function. (4 marks)

4. a. Using the solution  $u(x, t) = F(x + ct) + G(x - ct)$  of the wave equation  $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$

where  $F$  and  $G$  are arbitrary differentiable functions of one variable, solve the wave equation with initial data  $u(x, 0) = \frac{1}{1+x^2}$ ,  $\frac{\partial u}{\partial t}(x, 0) = 0$ ,  $-\infty < x < \infty$ , (6 marks)

b. Solve the wave equation with initial data

$$u(x, 0) = \frac{x}{1+x^2}, \quad \frac{\partial u}{\partial t}(x, 0) = -2xe^{-x^2} \quad (6 \text{ marks})$$

5. Consider the electrical cable running along the  $x$  - axis that is not well insulated from ground, so that leakage occurs along its entire length. Let  $V(x, t)$  and  $I(x, t)$  denote the voltage and current at point  $x$  in the wire at time  $t$ . These functions are related to each other by the system.

$$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} - RI, \quad \text{and} \quad \frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t} - GV,$$

Where  $L$  is the inductance,  $R$  is the resistance,  $C$  is the capacitance, and  $G$  is the leakage to ground. Show that  $V$  and  $I$  each satisfy

$$\frac{\partial^2 u}{\partial x^2} = LC \frac{\partial^2 u}{\partial t^2} + (RC + LG) \frac{\partial u}{\partial t} + RG u,$$

which is called the *telegraph equation*. (12 marks)

6. Solve the wave equation  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$  subject to the following boundary conditions:

$$u(0, t) = 0, \text{ for } t \geq 0 \text{ and } u(L, t) = 0, \text{ for } t \geq 0$$

and the initial conditions:

$$(x, 0) = f(x), \text{ for } 0 \leq x \leq L \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x), \text{ for } 0 \leq x \leq L$$

for the following data

(a).  $f(x) = \sin \frac{2\pi x}{L}$ ,  $g(x) = 0$ . (6 marks)

(b).  $f(x) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L}$ ,  $g(x) = 0$  (6 marks)