



**NATIONAL OPEN UNIVERSITY OF NIGERIA**  
**Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway. Jabi, Abuja.**  
**FACULTY OF SCIENCES**  
**DEPARTMENT OF MATHEMATICS**  
**September Examination, 2020\_1**

**Course Code: MTH 411**

**Course Title: Measure Theory and Integration**

**Credit Unit: 3**

**Time Allowed: 3 Hours**

**Instruction: Attempt Number One (1) and Any Other Four (4) Questions**

1. (a) Describe the greatest lower bound of the measures of all bounded open sets containing set  $G$ . **(1 mark)**  
(b) Let  $G_1, G_2$  be open sets such that  $G_1 \subseteq G_2$ , prove that  $m(G_1) \leq m(G_2)$ . **(5 marks)**  
(c) Let  $X$  be an arbitrary set. State the properties of the collection  $\Omega$  of subsets of  $X$  to be called an algebra. **(4 marks)**  
(d) Write on measurable function  $f: A \rightarrow [-\infty, +\infty]$  by stating the four conditions  $f$  must satisfy. **(4 marks)**  
(e) Distinguish between a measurable function and a Borel function using four examples. **(8 marks)**
  
2. Let  $(X, \mathcal{M})$  be a measurable space, and let  $\mu$  be a finitely additive measure on  $(X, \mathcal{M})$ . Show that  $\mu$  is a measure if either  
(i)  $\lim_k \mu(A_k) = \mu(\bigcup_k A_k)$  holds for each increasing sequence  $\{A_k\}$  of sets that belong to  $\mathcal{M}$ . Or  
(ii)  $\lim_k \mu(A_k) = 0$  holds for each decreasing sequence  $\{A_k\}$  of sets that belong to  $\mathcal{M}$  and satisfy  $\bigcap_k A_k = \emptyset$ . **(12 marks)**
  
3. (a) When is  $S: X \rightarrow \mathbb{R}$  a simple function? **(2 marks)**  
(b) State Beppo Levi's theorem. **(4 marks)**  
(c) Let  $(X, \mathcal{M})$  be a measurable space, let  $A$  be a subset of  $X$  that belongs to  $\mathcal{M}$ , and let  $f$  and  $g$  be  $[-\infty, +\infty]$ -valued measurable functions on  $A$ . Show that  $f \vee g$  and  $f \wedge g$  are measurable. **(6 marks)**
  
4. (a) State (i) Monotone Convergence theorem. **(3 marks)**  
(ii) Dominated Convergence theorem. **(4 marks)**  
(b) Let  $(X, \mathcal{M}, \mu)$  be a measure space, and let  $f$  and  $g$  be extended real-valued functions on  $X$  that are equal almost everywhere. If  $\mu$  is complete and if  $f$  is measurable, explain what is meant by  $g$  is measurable. **(5 marks)**
  
5. Let  $(X, \mathcal{M}, \mu)$  be a measure space. Explain the properties that hold almost everywhere. **(12 marks)**
  
6. (a) Define equivalent functions. **(3 marks)**  
(b) Let  $\{f_n\}$  be a sequence of measurable functions.  $f_n: X \rightarrow \mathbb{C}$  a.e. Suppose that  $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$ . Show that  $\sum_{n=1}^{\infty} f_n(x)$  converges to  $f(x)$  a.e on  $X$ . **(9 marks)**