



NATIONAL OPEN UNIVERSITY OF NIGERIA
Plot 91, Cadastral Zone, Nnamdi Azikwe Expressway, Jabi, Abuja.

FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
2023_2 EXAMINATIONS

Course Code: MTH 422
Course Title: Partial Differential Equations
Credit Unit: 3
Time Allowed: 3 Hours
Instruction: Answer Question Number one (1) and any Other Four (4) Questions

1. (a) In the following decide whether the given partial differential equations and boundary or initial conditions are linear or nonlinear, and, if linear, whether they are homogeneous or nonhomogeneous. Determine the order of the partial differential equation.
- i. $u_{xx} = u_t, u(x, 0) = 1, u(x, 1) = 0$
 - ii. $u_t u_x + u_{xt} = 2u, u(0, t) + u_x(0, t) = 0$
 - iii. $u_{xx} + e^t u_{tt} = u \cos x, u(x, 0) + u(x, 1) = 0$ **(2 marks each)**
- (b) Give any 5 classical examples of partial Differential equations **(2 marks each)**
- (c) Verify that the function $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of the three dimensional Laplace equation.
- $$u_{xx} + u_{yy} + u_{zz} = 0 \quad \textbf{(3 marks)}$$
- (d) Solve the given equation by the method of characteristic curves
- $$\frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0 \quad \textbf{(3 marks)}$$
2. (a) Using the following solution $u(x, t) = F(x + ct) + G(x - ct)$ solve the wave equation
- $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
- for the data $u(x, 0) = e^{-x^2}, \frac{\partial u}{\partial t}(x, 0) = 0, -\infty < x < \infty.$ **(6 marks)**
- (b) Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, u(0, t) = 0, \text{ for } t \geq 0 \text{ and } u(L, t) = 0, \text{ for } t \geq 0$ and initial conditions, $u(x, 0) = f(x), \text{ for } 0 \leq x \leq L$
and $\frac{\partial u}{\partial t}(x, 0) = g(x) \text{ for } 0 \leq x \leq L$ for the data:
- $$f(x) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L}, g(x) = 0 \quad \textbf{(6 marks)}$$

3. (a). Find the values of λ^2 for which the boundary value problem:

$$\frac{d^2u}{dx^2} + \lambda^2u = 0, \quad 0 < x < \frac{\pi}{2}$$

$$u(0) = 0 \text{ and } \int_0^{\frac{\pi}{2}} u(t)dt = 0, \text{ has nontrivial solutions} \quad \textbf{(6 marks)}$$

- (b). Let $v(x)$ be the steady state solution to the boundary value initial value problem

$$\frac{\partial^2u}{\partial x^2} + r = \frac{1}{k} \frac{\partial u}{\partial t} \quad 0 < x < a, \quad t > 0$$

$$u(0, t) = T_0$$

$$\frac{\partial u}{\partial x}(a, t) = 0$$

where r is a constant. Find and solve the boundary value problem for the steady state solution. **(6 marks)**

4. Solve the Laplace equation in the square $0 < x, y < \pi$ with the boundary conditions

$$\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = 0 \quad 0 < x < \pi, \quad 0 < y < \pi$$

$$u(0, y) = 0$$

$$u(x, y) = 0$$

$$u(x, 0) = 0$$

$$u(x, \pi) = 1$$

(12 marks)

5. Consider the following eigenvalue problem on the interval $(0, 1)$:

$$u''(x) + 2u'(x) - u(x) + \lambda(x + 1)^2e^{-2x}u(x) = 0$$

$$u'(0) = 0$$

$$u'(1) = 0$$

- (a). Explain the meaning of eigenvalue problem **(2 marks)**
 (b). Show that the eigenvalue problem is not Sturm-Liouville type **(3 marks)**
 (c). Multiply the equation above by e^{2x} to obtain a Sturm-Liouville problem. Identify $p(x), q(x),$ and $\sigma(x)$ **(3 marks)**
 (d). Suppose by Rayleigh quotient the leading eigenvalue is positive, that is $\lambda_1 > 0$, Find an upper bound for the leading eigenvalue **(4 marks)**

6. (a). Suppose that u_1 and u_2 are solutions of $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$, show that $c_1u_1 + c_2u_2$ is also a solution, where c_1 and c_2 are constants. **(2 marks)**

- (b). Solve the normalized wave equation

$$\frac{\partial^2u}{\partial t^2} = \frac{\partial^2u}{\partial x^2} \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0$$

$$u(x, 0) = \sin x, \quad \frac{\partial u}{\partial x}(x, 0) = \sin x$$

(10 marks)