

FACULTY OF SCIENCES  
DEPARTMENT OF MATHEMATICS  
2023 2 EXAMINATIONS

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Course Code: MTH 312

Course Title: Group and Rings

Credit Unit: 3

Time Allowed: 3 Hours

Total: 70 Marks

Instruction: Answer Question Number One and Any Other Four Questions

1. Let  $G$  be a group. An isomorphism  $\varphi: G \rightarrow G$  is called an automorphism of  $G$  and the set of all automorphisms of  $G$  is denoted by  $Aut(G)$ .
  - (a) Show that  $Aut(G)$  is a group under composition of functions. (4 marks)
  - (b) Let  $a \in G$  be fixed. Define the map  $\theta_a: G \rightarrow G$  by  $\theta_a(x) = ax^{a-1}$  for all  $x \in G$ . Show that  $\theta_a \in Aut(G)$ . (4 marks)
  - (c) Let  $Inn(G)$  be the set of all inner automorphisms of  $G$ . Show that  $Inn(G)$  is a normal subgroup of  $Aut(G)$ . (4 marks)
  - (d) For  $x, y \in G$ , we say that  $y$  is conjugate to  $x$ , if  $\exists a \in G$  such that  $y = \theta_a(x)$ .
    - i. Show that  $\sim$  is an equivalence relation on  $G$ . (4 marks)
    - ii. Let  $H < G$ . Show that set  $\theta_a(x) | x \in H$  is a subgroup of  $G$ . (3 marks)
  - (e) Show that  $(nZ, +, \cdot)$  is a ring where  $n \in Z$ . (3 marks)
2.
  - (a) Obtain the subgroup of  $S_4$ , to which  $Z_4$  is isomorphic. Is  $Z_4 = A_4$ ? (6 marks)
  - (b) Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  between mappings such that  $g \circ f$  is injective. Show that  $f$  must be injective but  $g$  need not be injective. (6 marks)
3.
  - (a) Find the remainder when  $a(x) = x^6 + 3x^6 - 3x + 2$  is divided by  $b(x) = 3x^2 + 2x - 3$  in  $Z_7(x)$ . (6 marks)
  - (b) Find the  $g.c.d$  of  $a(x) = x^6 + 3x^6 - 3x + 2$  and  $b(x) = 3x^2 + 2x - 3$  in  $Z_7(x)$ . (6 marks)
4.
  - (a) Find the polynomials  $r(x)$  and  $s(x)$  such that  $d(x) = a(x) \cdot r(x) + b(x) \cdot s(x)$ . (6 marks)
  - (b) Show that the functions  $f(x) = x^2$  and  $g(x) = x^3$  commute with each other under composition. (6 marks)
5.
  - (a) Show that a group  $G$  order 255 ( $3 \times 5 \times 17$ ) has either 1 or 51 Sylow 5 subgroups. (6 marks)
  - (b) How many Sylow 3-subgroups can it have? (6 marks)
6. Consider the set  $M_2(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$ . Show that  $M_2(R)$  is a ring with respect to addition and multiplication of matrices. (12 marks)