NATIONAL OPEN UNIVERSITY OF NIGERIA
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FACULTY OF SCIENCES
DEPARTMENT OFMATHEMATICS

Course Code:	**MTH382**
Course Title:	**Mathematical methods IV**
Credit Unit:	**3**
Time Allowed:	**3 Hours**
Total:	**70 Marks**
Instruction:	**Answer Question Number One and any Other Four Questions**

1. (a) Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(i) $dy/dx = xy^{1/2}$; $y = \frac{1}{16}x^4$ (ii) $y'' - 2y' + y = 0$; $y = xe^x$ **(2 marks)**

(b) In the problem below, verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval ***I*** of definition for each solution

(i) $\frac{dP}{dt} = P(1 - P)$; $P = \frac{c_1 e^t}{1 + c_1 e^t}$ (ii) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 12x^2$;
$y = c_1 x^{-1} + c_2 x + c_3 x \operatorname{In} x + 4x^2$ **(2 marks)**

(c) In the problem bellows, $y = 1/(1 + c_1 e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

(i) $y(0) = -\frac{1}{3}$ (ii) $y(-1) = 2$

(iii) Solve the given differential equation.

$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ **(3 marks)**

(d) Find the first three picard approximations for the initial value problem

(i) $y' = t + y$, $y(0) = 1$

(ii) Consider the initial value problem

$$y' = t + y \quad y(0) = 1.$$

For $n \geq 1$, find the *nth* Picard approximation and determine the limiting function $y = \lim_{n \to \infty} y_n$. Show that this function is a solution and, in fact, the only solution. **(2 marks)**

(iii) Show that the following differential equations have unique solutions on all of $\mathbb{R}$

I. $y' = e^{\sin ty}$, $y(0) = 0$ **II.** $y' = |ty|$, $y(0)\ 0$ **(2 marks)**

(e) (i) Write the corresponding integral equation for the initial value problem. $y' = \frac{t-y}{t+y}$ $y(0) = 1$ **(1 mark)**

(ii) Find the first *n* Picard approximations for the initial value problems

$y' = ty$, $y(1) = 1$, $n = 3$ **(1 mark)**

(iii) Which of the following initial value problems are guaranteed a unique solution by Picard's theorem? Explain.

I. $y' = 1 + y^2$, $y(0) = 0$

II. $y' = \sqrt{y}$, $y(1) = 0$

III. $y' = \sqrt{y}$, $y(0) = 1$

IV. $y' = \frac{t-y}{t+y}$, $y(0) = -1$

V. $y' = \frac{t-y}{t+y}$, $y(1) = -1$ **(4 marks)**

(f) (i) Derive $\Gamma(½)$ **(3 mark)**

(ii) Prove Stirling's approximation $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ for large *n* **(1 mark)**

(iii) Prove the result $\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2} dy$ **(1 mark)**

2. (a) Show that if *n* is a positive integer

$$\Gamma(n, x) = (n-1)!e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$$ **(2 marks)**

(b) (i) Show that $F(m, -m, ½ ; (1-x)/2) = T_m(x)$. **(3 marks)**

(ii) Prove the result 2(b)(i) **(3 marks)**

(c) (i) Show that setting $x = z/b$ in the hypergeometric equation, and letting $b \to \infty$, yields the confluent hypergeometric equation. **(2 marks)**

(ii) Prove the result $M(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{tx} t^{a-1}(1-t)^{c-a-1} dt$ **(2 marks)**

3. (a) (i) Show that $T_n'(x) = nU_{n-1}(x)$, **(3 marks)**

$$(1 - x^2)U_n'(x) = xU_n(x) - (n+1)T_{(n+1)}(x)$$

(ii) Find the general solution of $x^2y'' + xy' + (x^2 - 1/4)y = 0$. **(3 marks)**

(b) If n is an integer, show that $Y_{n+\frac{1}{2}}(x) = (-1)^{n+1}J_{-n-1/2}(x)$. **(3 marks)**

(c) Show that $f(x) = J_v(\propto x)$ satisfies $x^2f'' + xf' + (\propto^2x^2 - v^2)f = 0$ **(3 marks)**

4. (a) Evaluate the integral

$$\int_a^b J_v^2(\propto x)x\,dx$$ **(4 marks)**

(b) Given that $J_{1/2}(x) = (2/\pi x)^{1/2}\sin x$ and that $J_{-1/2}(x) = (2/\pi x)^{1/2}\cos x$, express $J_{3/2}(x)$ and $J_{-3/2}(x)$ in terms of trigonometry functions. **(4 marks)**

(c) Use the generating function to prove, for integer v, the recurrence relation

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$ **(4 marks)**

5. (a) Show that for integer n the Bessel function $J_v(x)$ is given by

$$J_v(x) = \frac{1}{n}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$ **(3 marks)**

(b) (i) Show that the l *th* spherical Bessel function is given by

$$f_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l f_o(x),$$ **(3 marks)**

where $f_l(x)$ denotes either $j_l(x)$ or $n_l(x)$

(ii) Use the Wrunskian method to find a closed-form expression for $Q_0(x)$. **(2 marks)**

(c) (i) Use Rodrigues' formula to show that

$$I_l = \int_{-1}^{1} P_l(x)P_l(x)dx = \frac{2}{2l+1}.$$ **(2 marks)**

(ii) Prove directly that the Legendre polynomials $P_l(x)$ are mutually orthogonal over the interval $-1 < x < 1$. **(2 marks)**

6. (a) For

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0,$$

find;

(i) the solution that takes the value $2y + 1$ on the line $x = 1$, and

(ii) a solution that has the value **4** at the point (1, 1) **(4 marks)**

(b) Find the general solution of

$x\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} - 2u = 0$ **(4 marks)**

(c) Find the general solution of

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 3x$$

Hence find the most general particular solution

(i) which satisfies $u(x, 0) = x^2$ and

(ii) which has the value $u(x, y) = 2$ at the point (1, 0). **(4 marks)**