



NATIONAL OPEN UNIVERSITY OF NIGERIA
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FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS
October Examination 2019

Course Code: MTH 412

Course Title: Functional Analysis II

Credit Unit: 3

Time Allowed: 3 Hours

Instruction: Answer Question number One and Any Other Four Questions

1. (a) Define (i) a norm on a linear space over a scalar field $K = (\mathbb{R} \text{ or } \mathbb{C})$. **(3 marks)**
(ii) l_∞ - space. **(3 marks)**
(b) Let $\| \cdot \|$ be a norm defined on a linear space X . If $\rho: X \times X \rightarrow \mathbb{R}$ is defined for arbitrary $x, y \in X$ by $\rho(x, y) = \|x - y\|$.
Show that ρ is a metric on X and so (X, ρ) is a metric space. **(8 marks)**
(c) Find the adjoint of $G: C^3 \rightarrow C^3$ defined by
(i) $G(x, y, z) = [3x + (1 - i)y, (2 + i)x - 3iz, 4ix + (3 - 2i)y - 3z]$; **(4 marks)**
(ii) $F(x, y, z) = [-2x + (3 + i)y, (4 - i)x + 2iz, 5ix + (1 - 2i)y - 7z]$. **(4 marks)**
2. (a) Let x and v be vectors in R^n , the line L through x in the direction of v given by
 $L = \{x + av: a \in \mathbb{R}\}$. Show that L is a convex set. **(6 marks)**
(b) Consider vectors $x = (1, 4, -5, -3)$ and $y = (-2, 1, 2, 3)$ in R^4 .
Find $\|x\|^2$ and $\|y\|^2$. **(6 marks)**
3. (a) State Bessel's inequality. **(4 marks)**
(b) Let (X, ρ) be a complete metric space and let $E \subset X$. Show that (E, ρ_E) is complete if and only if it is closed, where ρ_E is the subspace metric induced by ρ . **(8 marks)**

4. (a) Suppose $\{u_1, u_2, \dots, u_n\}$ is an orthogonal set of vectors. Show that

$$\|u_1 + u_2 + \dots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \dots + \|u_n\|^2. \quad (6 \text{ marks})$$
- (b) Give three (3) examples of incomplete normed linear spaces. **(6 marks)**
5. (a) State Riesz Representation Theorem. **(4 marks)**
- (b) Give an example of a linear space that is not a Hilbert space with its norm. **(4 marks)**
- (c) Let S consists of the following three vectors in \mathbb{R}^3 :

$$u_1 = (2, 4, 2), u_2 = (4, 2, -8), u_3 = (6, -4, 2).$$
- Verify that the vectors are orthogonal and hence they are linearly independent. **(4 marks)**
6. Let E be an inner product space, U is a subspace of E and x an arbitrary vector in E . If there exists a vector u^* in U such that $\|x - u^*\| = \inf_{u \in U} \|x - u\|$. Show that u^* is unique. In fact, $u^* \in U$ is a unique minimizing vector if and only if $(x - u^*) \perp U$. **(12 marks)**